Wightman Distributions & Reconstruction.

Let Φ be field op in WQFT acting on $S = S(\mathbb{R}^{1,D-1})$

$$\phi : S \longrightarrow O(\mathbb{H})$$

we assume $\phi(f)$ is self adjoint for real valued f.

Hence usually $\phi(f)^* = \phi(\bar{f})$

then for $f_1, \dots, f_N \in S$

define

$$W_N : S \times S \times \dots \times S \longrightarrow \mathbb{C}$$

$$W_N(f_1, \dots, f_N) := \langle \Omega, \phi(f_1) \dots \phi(f_N) \Omega \rangle$$

is multilinear and separately cts $\xRightarrow[\text{theorem}]{\text{nuclear}}$ (Schwartz) W_N is distribution on $S((\mathbb{R}^D)^N) = S(\mathbb{R}^{DN})$.

also call these <u>correlation functions</u>, <u>V.E.V's</u>, or <u>Wightman Distributions</u>

⊓ <u>Aside:</u>

Nuclear theorem of Schwartz refers to the generalization of schwartz kernel theorem by Grothendieck to nuclear vector spaces. ⊔

Thm

The WD's W_N for each field ϕ in WQFT satisfy:

W_N is tempered.

WD1 (covariance)

W_N is Poincaré invariant

$$W_N(f) = W_N((\Lambda, a) f)$$

$$\forall (\Lambda, a) \in P.$$

WD2 (Locality)

$1 \leq j < N$

$$W_N(x_1, \ldots, x_j, x_{j+1}, \ldots, x_N)$$

$$= W_N(x_1, \ldots, x_{j+1}, x_j, \ldots, x_N).$$

if $(x_j - x_{j+1})^2 < 0$

space-like seperated.

Here formally think of ϕ as coordinates but really it is the support of the functions x_j and x_{j+1}.

WD3 Spectrum

for each N $\exists\, M_N \in S'(\mathbb{R}^{D(N-1)})$

supported in $(\overline{C}_+)^{N-1} \subset \mathbb{R}^{D(N-1)}$

s.t.

$$W_N(x_1, \ldots, x_N) = \int_{\mathbb{R}^{D(N-1)}} M_N(p)\, e^{i \sum p_j \cdot (x_{j+1} - x_j)}\, dp$$

$$p = (p_1, \ldots, p_{N-1}) \in \mathbb{R}^{D(N-1)}$$

WD4 Positive Definiteness

For any sequence $f_N \in S(\mathbb{R}^{DN})$

one has for all $m \in \mathbb{N}$

$$\sum_{M,N=0}^{k} W_{M+N}(\overline{f_M} \otimes f_N) \geq 0$$

Notation!

denote W_N and ϕ as functions in order to capture properties of the support.

Proof

W1 - W4

+ a Lemma.

$\boxtimes$

thm

Given a sequence of $\{W_N\} \in S'(\mathbb{R}^{DN})$

obeying WD1-WD4

$\exists$ WQFT s.t. W_N are the WD's.

Proof Ingredients:

1. Let $\underline{S} := \bigoplus_{N=0}^{\infty} S(\mathbb{R}^{DN})$ algebra

+ functional $\Rightarrow$ bilinear form

2. Construct ideal J with degenerate elements

take quotient $\frac{\underline{S}}{J}$ which is a Pre-Hilb

$\overset{\text{complete}}{\Longrightarrow}$ $\mathcal{H}$ hilbert space.

4. $\phi(f)$ is fixed by, Ω is the multiplication induced, unit of $\underline{S}$.

5. Use WD1-4 to show W1-4.

Analytic Continuation & Wick Rotation

Types of analytic continuation:

1. holomorphic functions

$$\begin{array}{ccc} U & \overset{\text{connected domain}}{\hookrightarrow} & V \\ \text{\rotatebox{90}{$\subset$}} & & \text{\rotatebox{90}{$\subset$}} \\ \mathbb{C} & & \mathbb{C} \end{array}$$

famously Riemann zeta $\zeta(s)$ $\mathrm{Re}(s)>1$ to $\mathbb{C}$ and $\zeta(-1) = -\frac{1}{12}$

2. real analytic function $g: \underset{\mathbb{R}^n}{\overset{\cap}{W}} \to \mathbb{C}$

continue to open in $\mathbb{C}^n$ by power series.

3. Laplace transform:

E.g. $u: \mathbb{R}_+ \to \mathbb{C}$ polynomially bounded cts function.

Then

$$\mathcal{L}(u)(z) := \int_0^\infty u(t)\, e^{itz}\, dt, \quad \mathrm{Im}\, z \in \mathbb{R}_+$$

is hol on $U = \mathbb{R} \times \mathbb{R}_+ \subset \mathbb{C}$

with

$$\lim_{y \to 0} \left(L(u)(x+iy) \right) = g(x) := \int_0^\infty u(t)\, e^{itx}\, dt$$

called boundary value.

4. Distributions

Let $C \subset \mathbb{R}^n$ convex cone and

dual $C' = \{ p \in \mathbb{R}^n : p \cdot x \geq 0 \ \forall x \in C \}$

and C' non empty interior C^o

Let $\mathcal{T} := \mathbb{R}^n \times (C^o)$

open tube in $\mathbb{C}^n$.

In particular forward cone

$C = C_+ \quad \mathbb{R}^D = \mathbb{R}^{1,D-1}$

$C = C'$ and

$C^o = \{ x \in \mathbb{R}^{1,D-1} : x^2 = \langle x, x \rangle > 0, x^0 > 0 \}$

Thm

For every $T \in \mathcal{S}(\mathbb{R}^n)'$

whose Fourier transform has its support in $\mathbb{C}$ $\exists$ holomorphic F on $J \subset \mathbb{C}^n$ with

- $|F(z)| \leq c(1+|z|)^k (1+d^{-1}(z, \partial J))^m$ for suitable c, k, m.

i.e. a boundedness condition

- T is the boundary value of F, i.e. $\lim_{t \to 0} \int_{\mathbb{R}^n} f(x) F(x+ity)\,dx = T(f)$

Analytic Continuation of Wightman functions

Given $\phi : \mathcal{S}(\mathbb{R}^{1,D-1}) \longrightarrow \mathcal{O}(\mathbb{H})_{-}$, W_N

can be analytically continued

to $U \subset \mathbb{C}^{DN} \cong \mathbb{R}^{DN} \otimes \mathbb{C}$

where the Minkowski inner product

is continued to a complex bilinear fun

$$\langle z, w \rangle = z^0 w^0 - \sum_{j=1}^{D-1} z^j w^j .$$

With this we can identify

$\mathbb{R}^D$ with $E := \{(it, x^1, \ldots, x^{D-1}) \in \mathbb{C}^D$

$\mid (t, x^1, \ldots, x^{D-1}) \in \mathbb{R}^D\}$

which are "Euclidean points."

Theorem

W_N has an analytic continuation to $\mathcal{J}_N^{pe}$ which contains all ~~non~~ non coincident points of E^N

$$\mathcal{J}_N^{pe} := \bigcup \{ {}^{\sigma}\mathcal{J}_N^{e} : \sigma \in S_N \}$$

$$\mathcal{J}_N^{e} = \{ \Lambda(\mathcal{J}_N) : \Lambda \in L(\mathbb{C}) \}$$

$L(\mathbb{C})$ is proper complex Lorentz group.

and then ${}^{\sigma}\mathcal{J}_N^{e}$ is given by $\mathcal{J}_N^{e}$ with permuted coordinates

non coincident is $E^N \setminus \Delta$

Def

Schwinger functions

$$S_N := W_N|_{E^N \setminus \Delta}.$$

let $\Theta : E \to E$

$$(it, x^1, \dots, x^{D-1}) \mapsto (-it, x^1, \dots, x^{D-1})$$

time reflection

action on

$$\mathcal{S}(\mathbb{R}^{DN}) = \{ f : E^N \to \mathbb{C} \quad f \in \mathcal{S}(E^N) \quad \mathrm{supp}(f) \subset Q_+^N \}$$

where

$$Q_+^N := \{ (x_1, \dots, x_N) : x_j = (it_j, x^1, \dots, x^{D-1}), \quad 0 < t_1 < \dots < t_N \}$$

i.e. time ordered points.

$$\Theta : \mathcal{S}_+(\mathbb{R}^{DN}) \to \mathcal{S}(\mathbb{R}^{DN})$$

$$\Theta f(x_1, \dots, x_N) := \bar{f}(\Theta x_1, \dots, \Theta x_N).$$

thm

Schwinger functions satisfy:

S1 (Covariance)

$$S_N(gx_1, \dots, gx_N) = S_N(x_1, \dots, x_N)$$

for Euclidean transformations

$(a, R) \in \mathbb{R}^D \times SO(D)$

or Spin(D)

S2 (Locality)

$$S_N(x_1, \dots, x_N) = S_N(x_{\sigma(1)}, \dots, x_{\sigma(N)})$$

for any $\sigma \in S_N$.

S3 (Reflection Positivity)

$$\sum_{M,N} S_{M+N}(\Theta f_M \otimes f_N) \geq 0$$

These are called Osterwalder-Schrader axioms.

Reconstruction